$SU(2)$ slave-boson formulation of spin nematic states in $S=\frac{1}{2}$ frustrated ferromagnets

Ryuichi Shindou and Tsutomu Momoi
Condensed Matter Theory Laboratory, RIKEN, 2-1 Hirosawa, Wako, Saitama 351-0198, Japan
(Received 5 June 2009; revised manuscript received 27 July 2009; published 17 August 2009)

An $SU(2)$ slave-boson formulation of bond-type spin nematic orders is developed in frustrated ferromagnets, where the spin nematic states are described as the resonating *spin-triplet* valence bond (RVB) states. The $\boldsymbol{d}$ vectors of spin-triplet pairing ansatzes play the role of the directors in the bond-type spin-quadrupolar states. The low-energy excitations around such spin-triplet RVB ansatzes generally comprise the (potentially massless) gauge bosons, massless Goldstone bosons, and spinon individual excitations. Extending the projective symmetry-group argument to the spin-triplet ansatzes, we show how to identify the number of massless gauge bosons efficiently. Applying this formulation, we next (i) enumerate possible mean-field solutions for the $S=\frac{1}{2}$ *ferromagnetic* J_1-J_2 Heisenberg model on the square lattice, with ferromagnetic nearest neighbor J_1 and competing antiferromagnetic next-nearest neighbor J_2 and (ii) argue their stability against small gauge fluctuations. As a result, two stable spin-triplet RVB ansatzes are found in the intermediate coupling regime around $J_1:J_2 \simeq 1:0.4$. One is the Z_2 Balian-Werthamer (BW) state stabilized by the Higgs mechanism and the other is the $SU(2)$ chiral p-wave (Anderson-Brinkman-Morel) state stabilized by the Chern-Simon mechanism. The former Z_2 BW state in fact shows the same bond-type spin-quadrupolar order as found in the previous exact diagonalization study [Shannon *et al.*, Phys. Rev. Lett. **96**, 027213 (2006)].

DOI: 10.1103/PhysRevB.80.064410 PACS number(s): 75.10.Jm, 71.10.Pm, 75.40.Gb

I. INTRODUCTION

Recent theoretical progress has revealed that a certain class of frustrated magnets[1–10] shows spin nematic states[1,11] as their magnetic ground states, where the spin-quadratic tensor, $K_{jl,\mu\nu} \equiv \langle S_{j\mu} S_{l\nu}\rangle - \frac{\delta_{\mu\nu}}{3}\langle \boldsymbol{S}_j \cdot \boldsymbol{S}_l\rangle$ with $(\mu,\nu=1,2,3)$, exhibits a long-range order, while the spin moment $\langle S_{j\mu}\rangle$ remains disordered. Such spin nematic states can be classified into the chiral type (p-nematic) and nonchiral type (n-nematic) states,[1] according to the parity of the spin-quadratic tensor. Namely, the antisymmetric quadratic tensor $P_{jl,\lambda} \equiv \epsilon_{\lambda\mu\nu} K_{jl,\mu\nu}$ is nothing but the the vector chirality, while the symmetric part—nonchiral one—plays the role of the spin-quadrupolar moment, $Q_{jl,\mu\nu} \equiv \frac{1}{2}(K_{jl,\mu\nu}+K_{jl,\nu\mu})$. The latter ordered state is a spin analog of the nematic state well known in liquid crystals,[12] where the order parameter is characterized by the so-called "director vector" $\boldsymbol{d}(\boldsymbol{r})$ in the form

$$Q_{\mu\nu}(\boldsymbol{r}) = d_\mu(\boldsymbol{r}) d_\nu(\boldsymbol{r}) - \frac{1}{3}\delta_{\mu\nu}|\boldsymbol{d}(\boldsymbol{r})|^2. \tag{1}$$

From this analogy, the spin-quadrupolar states are often dubbed simply as the "spin nematic" states.[1] Depending on how the spin-quadrupolar moments are microscopically organized, spin nematic states have two distinct classes; (i) site-type nematic states[11,13–19] and (ii) bond-type nematic states.[1–10] The former types of nematic orders are realized in the spin one bilinear-biquadratic model, $\mathcal{H}_{S=1} = \Sigma_{\langle ij\rangle}[J\boldsymbol{S}_i \cdot \boldsymbol{S}_j + K(\boldsymbol{S}_i \cdot \boldsymbol{S}_j)^2]$, where the quadrupolar moments constituted *at respective sites* exhibit the long-range order due to the strong biquadratic coupling.[11,13–15,17] Ground-state wave functions of these site-type nematic states can be essentially factorized into decoupled "vacuums," which are defined on respective sites. Thus, their spin-wave theories[13–16,18,19] including low-energy effective theories[20] were well established. Namely, the elementary excitation around such a site-factorized vacuum is also given by a linear combination of bosons introduced at respective sites.

The simplest localized spin models which allow the second class of spin nematic states—bond-type nematic states—are the spin one-half frustrated ferromagnets,[1–10] which could be realized in a certain family of layered cuprates[21–25] and vanadates[26,27] and also in solid ^{3}He films.[28] For example, in $(CuX)LaNb_2O_7$ (X=Cl, Br),[21,22] Cu^{2+} ions, having a localized spin $\frac{1}{2}$, compose a square lattice, while the anion X^- locates at the center of the square instead of the bond center. As a result, the nearest-neighbor (NN) exchange interaction J_1 between the localized spins becomes ferromagnetic because of the Goodenough-Kanamori rule[29] while the next nearest-neighbor (NNN) interaction J_2 becomes antiferromagnetic; the model Hamiltonian is given by

$$\mathcal{H} = -J_1 \sum_{\langle j,l\rangle} \boldsymbol{S}_j \cdot \boldsymbol{S}_l + J_2 \sum_{\langle\langle j,l\rangle\rangle} \boldsymbol{S}_j \cdot \boldsymbol{S}_l \tag{2}$$

with $J_1, J_2 > 0$. The preceding exact diagonalization (ED) studies for this spin one half square-lattice J_1-J_2 model[5] indicated that the d-wave *bond-type* spin nematic order develops in the intermediate parameter region, $J_1 \simeq 2J_2$. Namely, strong ferromagnetic exchange interactions favor the spin-triplet valence-bond formations between two neighboring spin one halves, while, simultaneously, these two spin one halves try to change their partners quantum mechanically by way of the NNN antiferromagnetic exchange interactions. This leads to a kind of *resonating* spin-triplet valence-bond state, where the quadrupolar moment organized *at each neighbor bond* exhibits the following antiferro-type configuration with the uniform amplitude:

$$Q_{\langle j,j+\hat{x}\rangle,22} - Q_{\langle j,j+\hat{x}\rangle,11} = Q_{\langle j,j+\hat{y}\rangle,11} - Q_{\langle j,j+\hat{y}\rangle,22} > 0. \tag{3}$$

Similar bond-nematic order phases were also found in other frustrated ferromagnets, such as a zigzag spin chain[2,6,7] containing ferromagnetic J_1 and a triangular lattice multiple-spin-exchange model.[8–10]

In contrast to the site-type nematic states, however, when attempting to construct a mean-field description of these bond-nematic states [as well as their spin-wave theories], one could immediately reach a more fundamental question; how their ground-state wave functions themselves should be described? Namely, since a single spin one half at each site is supposed to participate *equally* in the spin-triplet formations on its four ferromagnetic bonds [in the square-lattice case], their ground-state wave functions are no longer described by any kind of "site-factorized wave functions."

In this paper, we will construct an $SU(2)$ slave-boson mean-field theory of the bond-type spin nematic states, which are described as the resonating valence bond (RVB) states of the *spin-triplet* bonds. After splitting the original spin operator into the bilinear of the spinon fields (fermions),[30–34] $S_{j\mu} \equiv \frac{1}{2} f^{\dagger}_{j\alpha}[\sigma_\mu]_{\alpha\beta} f_{j\beta}$, we first introduce the spin-triplet pairing ansatzes into the ferromagnetic exchange bonds as

$$E_{ij,\mu} \equiv \langle f^{\dagger}_{i\alpha}[\sigma_\mu]_{\alpha\beta} f_{j\beta}\rangle, \tag{4}$$

$$D_{ij,\mu} \equiv \langle f_{i\alpha}[i\sigma_2\sigma_\mu]_{\alpha\beta} f_{j\beta}\rangle, \tag{5}$$

where $\boldsymbol{D}_{ij}$ ($\boldsymbol{E}_{ij}$) describes the $\boldsymbol{d}$ vector of the spin-triplet pair condensation[35] ("spin-orbit" hopping integral). In fact, these two types of the $\boldsymbol{d}$ vectors, i.e., that in the particle-hole channel and in the particle-particle channel, precisely mimic the director vector $\boldsymbol{d}(\boldsymbol{r})$ of nematic states in liquid crystals [see Eq. (1)]; in the mean-field approximation, the quadrupolar order parameter is given by

$$\begin{aligned} Q_{jl,\mu\nu} = &-\frac{1}{2}\left(E_{jl,\mu}E^{*}_{jl,\nu} - \frac{1}{3}\delta_{\mu\nu}|\boldsymbol{E}_{jl}|^2\right) + \text{H.c.} \\ &-\frac{1}{2}\left(D_{jl,\mu}D^{*}_{jl,\nu} - \frac{1}{3}\delta_{\mu\nu}|\boldsymbol{D}_{jl}|^2\right) + \text{H.c.} \end{aligned} \tag{6}$$

Moreover, the vector chiral order parameter is given by the products between these two $\boldsymbol{d}$ vectors and their respective spin-singlet ansatzes in the form[36]

$$P_{jl,\lambda} = \frac{i}{2}(\chi_{jl}E^{*}_{jl,\lambda} - \chi^{*}_{jl}E_{jl,\lambda}) - \frac{i}{2}(\eta_{jl}D^{*}_{jl,\lambda} - \eta^{*}_{jl}D_{jl,\lambda}), \tag{7}$$

where χ_{jl} (η_{jl}) stands for the spinless hopping integral (spin-singlet pair condensation)[30–32]

$$\chi_{jl} \equiv \langle f^{\dagger}_{j\alpha} f_{l\alpha}\rangle, \quad \eta_{jl} \equiv \langle f_{j\alpha}[(-i)\sigma_2]_{\alpha\beta} f_{l\beta}\rangle. \tag{8}$$

Thus, one can naturally employ the spin-triplet slave-boson theory as a mean-field description of the spin nematic orders. In Sec. II, we will introduce an $SU(2)$ formulation of the spin-triplet mean-field ansatzes, where we extensively use the 2×2 matrix representation originally introduced by Affleck *et al.*,[37] instead of the usual Nambu vector. This representation [see Eqs. (10) and (12)] clearly dictates that the low-energy excitation around any spin-triplet RVB state generally consists of (gapless) Goldstone boson and (potentially gapless) gauge boson. It is widely known that the existence of the gapless gauge fluctuations is crucial to the instability of the starting mean-field ansatzes.[33,34,38] Thus, we will next argue the spin-triplet extension of the projective symmetry group (PSG) arguments. Without resorting to any microscopic calculations, this extension enables us to identify the number of the massless gauge bosons for any given mixed ansatz having both spin-triplet and spin-singlet link variables.

Armed with these general formulations, we study in Sec. III the ferromagnetic J_1-J_2 Heisenberg square-lattice model defined in Eq. (2), thereby finding two stable spin-triplet RVB ansatzes in the intermediate coupling region, $J_1 \simeq 2J_2$. One is the Balian-Werthamer- (BW-)type triplet pairing state[39] having the coplanar configurations of the $\boldsymbol{d}$ vector, $\hat{d}(k) \propto \hat{x}k_x + \hat{y}k_y$ while the other is the chiral p-wave state[40] having its $\boldsymbol{d}$ vector all pointing in the same direction $\hat{d}(k) \propto \hat{z}(k_x + ik_y)$ [see Fig. 2(b)]. The PSG arguments indicate that, in general, all the nonmagnetic (gauge) excitations in the BW state have finite Higgs mass. Thus, this ansatz—Z_2 BW state—is stable against any type of small gauge fluctuations. On the other hand, the chiral p-wave state does not break any of the $SU(2)$ gauge symmetry. Instead, it breaks the time-reversal symmetry and all the mirror symmetries. As a result, nonmagnetic (gauge) bosons are endowed by the Chern-Simon term with the topologically induced mass. Thus, this $SU(2)$ chiral p-wave state is also stable against any small gauge fluctuation. Though both the BW and chiral p-wave states exhibit spin-quadrupolar orders, the BW state especially shows the same configuration of quadrupolar moments as the bond-type spin nematic order found in Ref. 5. Hence, we further discuss possible experimental features of this BW state, mainly focusing on its magnetic excitations.

Sec. IV is devoted to the summary and open issues. The relation between our Z_2 BW state and the time-reversal topological insulator recently discussed in the various literatures[42–46] is briefly mentioned. We also propose those combinations of the triplet and singlet ansatzes which describe the vector chiral order having *no* finite director vector,[4] i.e., $P_{jl,\mu} \neq 0$ and $Q_{jl,\mu\nu} = 0$. Those readers who want to make the $SU(2)$ slave-boson study in frustrated ferromagnets be a *controlled* analysis might as well consult the Appendix, where we describe the large N generalization of frustrated ferromagnetic spin models.

II. $SU(2)$ FORMULATION OF SPIN-TRIPLET RVB STATE

A. Matrix representation

The slave-boson formulation begins with describing the spin operator by the bilinear of fermion fields; $2S_{j\mu} \equiv f^{\dagger}_{j\alpha}[\sigma_\mu]_{\alpha\beta} f_{j\beta}$. The enlarged (fermion's) Hilbert space reduces to the physical (spin's) Hilbert space, provided that the following local constraints are strictly observed at each site:

$$f^{\dagger}_{j\alpha}[\sigma_3]_{\alpha\beta} f_{j\beta} = 1, \quad f^{\dagger}_{j\uparrow} f^{\dagger}_{j\downarrow} = f_{j\downarrow} f_{j\uparrow} = 0.$$

In the partition function, these local constraints are implemented as the coupling between the fermion (spinon) fields and the temporal $SU(2)$ gauge fields $a^{\nu}_{j,\tau}$ ($\nu = 1,2,3$),[30,31,33,34]

$$Z \equiv \int d\vec{a}_{\tau} d\Psi^{\dagger} d\Psi \exp\left[-\int_0^{\beta} d\tau \mathcal{L} \right],$$

$$\mathcal{L} \equiv \frac{1}{2}\sum_j \mathrm{Tr}\left[\Psi_j^{\dagger}\left(\partial_{\tau}\sigma_0 + \sum_{\nu=1}^{3} i a_{j,\tau}^{\nu}\sigma_{\nu} \right)\Psi_j \right] + \mathcal{H}, \quad (9)$$

where Ψ_j and $\Psi_j^{\dagger}$ stand for the 2×2 matrices

$$\Psi_j \equiv \begin{bmatrix} f_{j,\uparrow} & f_{j,\downarrow} \\ f_{j,\downarrow}^{\dagger} & -f_{j,\uparrow}^{\dagger} \end{bmatrix}, \quad \Psi_j^{\dagger} \equiv \begin{bmatrix} f_{j,\uparrow}^{\dagger} & f_{j,\downarrow} \\ f_{j,\downarrow}^{\dagger} & -f_{j,\uparrow} \end{bmatrix}. \quad (10)$$

The spin-Hamiltonian part $\mathcal{H}$ becomes quartic in the fermion field (Ψ field). Depending on the sign of the exchange interaction, we decompose this quartic term into the Stratonovich-Hubbard variables in two alternative ways

$$Z = \int dU^{\mathrm{sin}} dU^{\mathrm{tri}} d\vec{a}_{\tau} d\Psi^{\dagger} d\Psi \exp\left[-\int_0^{\beta} d\tau \mathcal{L} \right], \quad (11)$$

$$\begin{aligned}\mathcal{L} = {} & \frac{1}{2}\sum_j \mathrm{Tr}\left[\Psi_j^{\dagger}\left(\partial_{\tau}\sigma_0 + \sum_{\nu=1}^{3} i a_{j,\tau}^{\nu}\sigma_{\nu} \right)\Psi_j \right] \\ & - \frac{J_1}{4}\sum_{\langle jl\rangle} \{(-|\boldsymbol{E}_{jl}|^2 - |\boldsymbol{D}_{jl}|^2) + \mathrm{Tr}[\Psi_j^{\dagger} U_{jl,\mu}^{\mathrm{tri}} \Psi_l \sigma_{\mu}^{T}]\} \\ & - \frac{J_2}{4}\sum_{\langle\langle jl\rangle\rangle} \{(-|\chi_{jl}|^2 - |\eta_{jl}|^2) + \mathrm{Tr}[\Psi_j^{\dagger} U_{jl}^{\mathrm{sin}} \Psi_l]\}. \end{aligned} \quad (12)$$

Namely, the triplet and singlet link variables

$$U_{ij}^{\mathrm{sin}} \equiv \begin{bmatrix} \chi_{ij}^{*} & \eta_{ij} \\ \eta_{ij}^{*} & -\chi_{ij} \end{bmatrix}, \quad U_{ij,\mu}^{\mathrm{tri}} \equiv \begin{bmatrix} E_{ij,\mu}^{*} & D_{ij,\mu} \\ -D_{ij,\mu}^{*} & E_{ij,\mu} \end{bmatrix} \quad (13)$$

are introduced as the auxiliary fields for the ferromagnetic and antiferromagnetic bonds, respectively. This is simply because the sign of the ferromagnetic exchange interaction generally allows us to perform the Gaussian integration only over the $\boldsymbol{d}$ vectors in the excitonic/Cooper channel. In fact, this integration precisely reproduces the ferromagnetic exchange interaction

$$\begin{aligned} -4\boldsymbol{S}_j\cdot\boldsymbol{S}_l = {} & -\sum_{\mu=1}^{3} (f_{j\alpha}^{\dagger}[\sigma_{\mu}]_{\alpha\beta} f_{l\beta})(f_{l\gamma}^{\dagger}[\sigma_{\mu}]_{\gamma\delta} f_{j\delta}) \\ & - \sum_{\mu=1}^{3} (f_{j\alpha}^{\dagger}[\sigma_2\sigma_{\mu}]_{\alpha\beta} f_{l\beta}^{\dagger})(f_{l\gamma}[\sigma_{\mu}\sigma_2]_{\gamma\delta} f_{j\delta}) \end{aligned}$$

while that over the singlet variable leads to the antiferromagnetic exchange interaction[30–34]

$$4\boldsymbol{S}_j\cdot\boldsymbol{S}_l = -(f_{j\alpha}^{\dagger} f_{l\alpha})(f_{l\beta}^{\dagger} f_{j\beta}) - (f_{j\alpha}^{\dagger}[\sigma_2]_{\alpha\beta} f_{l\beta}^{\dagger})(f_{l\gamma}[\sigma_2]_{\gamma\delta} f_{j\delta}).$$

Thus, the slave-boson formulation of *mixed* Heisenberg magnets generally requires us to use the spin-triplet link variable $U_{jl,\mu}^{\mathrm{tri}}$ for every ferromagnetic bond and the spin-singlet link variable U_{jl}^{sin} for every antiferromagnetic bond. The saddle-point solutions of Eq. (12) lead to the coupled gap equations for these link variables, i.e., Eqs. (4), (5), and (8) whose right-hand sides are self-consistently given by these mean-fields themselves. In terms of $U_{jl,\mu}^{\mathrm{tri}}$ and U_{jl}^{sin} thus determined, the spin-quadrupolar moment and vector chirality are given by

$$-2Q_{jl,\mu\nu} = \mathrm{Tr}[U_{lj,\mu}^{\mathrm{tri}} U_{jl,\nu}^{\mathrm{tri}}] - \frac{\delta_{\mu\nu}}{3}\sum_{\lambda=1}^{3} \mathrm{Tr}[U_{lj,\lambda}^{\mathrm{tri}} U_{jl,\lambda}^{\mathrm{tri}}], \quad (14)$$

$$-2iP_{jl,\lambda} = \mathrm{Tr}[U_{lj}^{\mathrm{sin}} U_{jl,\lambda}^{\mathrm{tri}}]. \quad (15)$$

Comparing Eq. (12) with Eq. (15) we notice that the present J_1-J_2 model can have spin-quadrupolar order on ferromagnetic bonds but cannot have vector chirality on any links since $U_{lj}^{\mathrm{sin}} U_{jl,\lambda}^{\mathrm{tri}} = 0$. Within our formalism, a naive mean-field description of vector chiral orders becomes possible only in those spin models having either symmetric anisotropic exchange interactions or antisymmetric anisotropic one. In the next section, without making any distinction between the n-nematic states and p-nematic ones, we will widely call those mean-field ansatzes having both finite triplet ansatz and singlet ansatz as spin-triplet RVB states.

B. Low-energy excitations around spin-triplet RVB states

To see the low-energy excitations around the spin-triplet RVB ansatzes, let us first express the spin operator in terms of the 2×2 matrix representation,[37] $S_{j\mu} \equiv \frac{1}{4}\mathrm{Tr}[\Psi_j^{\dagger}\Psi_j\sigma_{\mu}^{T}]$. Namely, a spin rotation is described by an $SU(2)$ matrix, say h_j, applied from the left- (right-)hand side of $\Psi_j^{\dagger}$ (Ψ_j)

$$\Psi_j \to \Psi_j h_j^{T}, \quad \Psi_j^{\dagger} \to h_j^{*}\Psi_j^{\dagger}$$

while physical quantities are invariant under any *local* $SU(2)$ gauge transformation applied from the right- (left-)hand side of $\Psi_j^{\dagger}$ (Ψ_j)

$$\Psi_j \to g_j\Psi_j, \quad \Psi_j^{\dagger} \to \Psi_j^{\dagger} g_j^{\dagger},$$

$$\{U_{jl,\mu}^{\mathrm{tri}}, U_{jl}^{\mathrm{sin}}\} \to g_j\{U_{jl,\mu}^{\mathrm{tri}}, U_{jl}^{\mathrm{sin}}\} g_l^{\dagger}.$$

For example, both parts of the spin-quadratic tensor, Eqs. (14) and (15), are invariant under this local $SU(2)$ gauge transformation. In regard to these two symmetries, any spin-triplet mean-field ansatz is generally accompanied by two types of low-energy excitations: the magnetic ones (Goldstone bosons)[47] and the the nonmagnetic ones (gauge bosons).[30,31,33,34,37,48]

The former excitations are semiclassically described by the deformations of the $\boldsymbol{d}$ vectors around its mean-field configuration

$$U_{jl,\mu}^{\mathrm{tri}} \equiv \sum_{\nu=1}^{3} \bar{U}_{jl,\nu}^{\mathrm{tri}} R_{\nu\mu}\left(\frac{j+l}{2}, \tau \right) \quad (16)$$

for $\mu = 1,2,3$ with the 3×3 rotational matrix $\hat{R}(x,\tau)$. Such deformations cost infinitesimally small energy in spin models with spin-continuous symmetry, provided that the variation in the rotation is sufficiently slow in space and time. This type of deformations describe the Goldstone modes accompanying the spontaneous symmetry breaking.

In addition to this conventional excitation, a certain nonmagnetic (gauge) excitations also become massless, when our starting mean-field ansatz is invariant under a *continuous* gauge symmetry.[30,31,33,34] For example, assume that the invariant gauge group (IGG) contains the $U(1)$ gauge symmetry $\{e^{i\theta\sigma_3}|\theta\in[0,2\pi)\}$. Namely, our mean-field ansatz is invariant under any rotation around the 3 axis in the gauge space

$$e^{i\theta\sigma_3}\{\bar{U}^{\rm tri}_{jl,\mu},\bar{U}^{\rm sin}_{jl}\}e^{-i\theta\sigma_3}=\{\bar{U}^{\rm tri}_{jl,\mu},\bar{U}^{\rm sin}_{jl}\} \tag{17}$$

for $\mu=1,2,3$ and $\bar{a}^{\nu}_{j,\tau}=\delta_{\nu 3}\bar{a}^{3}_{j,\tau}$. Then, we can argue that the following nonmagnetic deformation also comprises the gapless excitation:

$$\{U^{\rm tri}_{jl,\mu},U^{\rm sin}_{jl}\}\equiv\{\bar{U}^{\rm tri}_{jl,\mu},\bar{U}^{\rm sin}_{jl}\}e^{ia_{jl}\sigma_3}, \tag{18}$$

$$a^3_{j,\tau}\equiv\bar{a}^3_{j,\tau}+a_0(j,\tau), \tag{19}$$

where a_{jl} relates to the spatial components of "gauge fluctuations" $a_\alpha(j,\tau)$ $(\alpha=1,\ldots,d)$ in the form

$$a_{jl}(\tau)=(j-l)_\alpha a_\alpha(j,\tau). \tag{20}$$

Specifically, one can expand the effective action in terms of these variations $a_\alpha(j,\tau)$ $(\alpha=0,1,\ldots,d)$, assuming these fluctuations to be much smaller than their units, $a_\alpha(j,\tau)\ll 2\pi$. Up to their quadratic order, the effective action generally reads as follows:

$$F_{\rm gauge}=\sum_{\alpha,\beta=0}^{d}\sum_{Q}M_{\alpha\beta}(Q)a_\alpha(Q)a_\beta(-Q)+\cdots, \tag{21}$$

$$a_\alpha(Q)=\frac{1}{\sqrt{N\beta}}\sum_{i\omega_n}\sum_{q}e^{iqj-i\omega_n\tau}a_\alpha(j,\tau) \tag{22}$$

with $Q=(q,i\omega_m)$. Then, taking into account the $U(1)$ gauge symmetry of the mean-field ansatz, one can specify the form of the $(d+1)\times(d+1)$ matrix $\hat{M}(Q)$, such that the quadratic part in Eq. (21) reduces to the $U(1)$ gauge-invariant form as in Eq. (23).

To see this, introduce the following local $U(1)$ gauge transformation in Eq. (12):

$$\Psi^\dagger_j(\tau)\to\Psi^\dagger_j(\tau)e^{i\theta_j(\tau)\sigma_3},$$

$$\Psi_j(\tau)\to e^{-i\theta_j(\tau)\sigma_3}\Psi_j(\tau),$$

where $\theta_j(\tau)$ varies slowly in space and time. Under this transformation, all changes in the link variables [Eq. (18)] are put into the transformation, $a_{jl}\to a_{jl}+\theta_l-\theta_j$ and $a_0\to a_0+\partial_\tau\theta$, due to the $U(1)$ symmetry in IGG. Thus the effective action around $Q\simeq 0$ is literally transformed as

$$F_{\rm gauge}\to\sum_{\alpha,\beta}\sum_{Q\simeq 0}M_{\alpha\beta}(Q)(a_\alpha+\partial_\alpha\theta)(Q)(a_\beta+\partial_\beta\theta)(-Q).$$

However, the free energy should have been invariant under any gauge transformation since gauge degrees of freedom can be absorbed into the integral variables Ψ fields. This requires that $\hat{M}(Q)$ must precisely reduce to zero at $Q=0$, so that the quadratic part of the action takes $U(1)$ gauge-invariant forms, e.g.,

$$F^{U(1)}_{\rm gauge}=\frac{1}{8\pi}\sum_{Q\simeq 0}\sum_{\alpha=0}^{d}\frac{1}{g_\alpha^2}f_\alpha(Q)f_\alpha(-Q)+\cdots, \tag{23}$$

where $f_\gamma\equiv\epsilon_{\alpha\beta\gamma}\partial_\alpha a_\beta$ stands for the field strength.[30,31,33,34] It is well known that this maxwell form does not suppress the gauge fluctuation efficiently. Especially, when the mean-field ansatz have its fermionic excitations fully gapped and when $d=2$, these massless gauge fluctuations destroy the mean-field ansatz itself,[33,34,38] apart from some exceptional cases.[49–54] Following the literature,[33] we call in this paper such spin-triplet mean-field ansatz as the gapped $U(1)$ [or $SU(2)$] state.

On the other hand, if the starting mean-field ansatz has no continuous IGG, like in Eq. (17), the local minimum condition imposed on mean-field ansatzes generally requires all the eigenvalues of $\hat{M}(Q)$ to be positive. Therefore, all the gauge fields have finite Higgs mass around any Q

$$F^{Z_2}_{\rm gauge}=\sum_{Q}\sum_{\alpha=0}^{d}\tilde{M}_\alpha(Q)\tilde{a}_\alpha(Q)\tilde{a}_\alpha(-Q)+\cdots \tag{24}$$

with $\tilde{M}_\alpha(Q)>0$. In contrast to the maxwell form discussed above, this finite Higgs mass suppresses any small gauge fluctuation completely. Hence the starting mean-field ansatz is always guaranteed to be (at least locally) stable. Such ansatzes are usually dubbed as the Z_2 state.

The efficient way to confirm the absence of the continuous IGG was introduced by Wen[33,48] where he pointed out the sufficient condition for its absence. We can extend his argument to the spin-triplet RVB states also. To see this, let us begin with the calculation of the $SU(2)$ flux defined on a plaquette by multiplying link variables along the closed loop in a regular sequence, where either $\bar{U}^{\rm sin}_{ij}$ or one of $\bar{U}^{\rm tri}_{ij,\mu}$ should be chosen on each link. For example, when the loop is given by a triangular path $i\to j\to k\to i$, one can have an $SU(2)$ flux by $\bar{U}_{ij}\bar{U}_{jk}\bar{U}_{ki}$, which always transforms in a gauge-covariant way

$$\bar{U}_{ij}\bar{U}_{jk}\bar{U}_{ki}\to g_i^\dagger\cdot\bar{U}_{ij}\bar{U}_{jk}\bar{U}_{ki}\cdot g_i$$

under $\Psi_j\to g_j\Psi_j$. As such, the *relative angle* subtended by two distinct $SU(2)$ fluxes derived from the same base site, such as $\bar{U}_{ij}\bar{U}_{jk}\bar{U}_{ki}$ and $\bar{U}_{ij}\bar{U}_{jl}\bar{U}_{li}$, contains nontrivial *gauge-independent* information, provided that the two triangular paths, $\langle ijk(i)\rangle$ and $\langle ijl(i)\rangle$, are different with each other. Note that, even out of the *same* triangular loop, we can have *two* distinct fluxes, when one of its three links has two different types of spin-triplet ansatzes, $\bar{U}^{\rm tri}_{ij,1}\neq\bar{U}^{\rm tri}_{ij,2}$. In this case, we should regard that $\bar{U}^{\rm tri}_{ij,1}\bar{U}^{\cdot\cdot}_{jk}\bar{U}^{\cdot\cdot}_{ki}$ and $\bar{U}^{\rm tri}_{ij,2}\bar{U}^{\cdot\cdot}_{jk}\bar{U}^{\cdot\cdot}_{ki}$ are two distinct fluxes obtained from the same base site i.

Having all $SU(2)$ fluxes thus obtained in hand, one can readily see that, **(i)** if two distinct $SU(2)$ fluxes obtained from the same base site are not collinear with each other, there is no continuous IGG in that mean-field ansatz. **(ii)** If all the distinct fluxes obtained from the same base site are pointed along one direction in the gauge space, say along the 3 axis, the ansatz could have a certain $U(1)$ gauge symmetry around this 3 axis, just like in Eq. (17). One can also confirm that,

(iii) the ansatz can be invariant under a certain $SU(2)$ gauge symmetry [so-called $SU(2)$ state], if all the $SU(2)$ fluxes are proportional to the unit matrix.

This "noncollinearity" argument of the $SU(2)$ fluxes concludes the (local) stability of each ansatz against gauge fluctuations very efficiently without resorting to any microscopic calculation. Thus, it substantially helps us to find a better spin-triplet mean-field ansatz as in the case of spin-single RVB ansatzes.[33,48]

III. J_1-J_2 FRUSTRATED FERROMAGNETIC SQUARE-LATTICE HEISENBERG MODEL

In this section, we will apply the spin-triplet slave-boson mean-field formulation onto the spin-$\frac{1}{2}$ J_1-J_2 mixed Heisenberg model (2) on the square lattice with ferromagnetic NN J_1 and antiferromagnetic NNN J_2. As was described in the previous section, we always decompose the ferromagnetic NN bond into the spin-triplet ansatz and the antiferromagnetic NNN bond into the spin-singlet ansatz.

A. Mean-field solutions

To be specific, we have numerically studied the various local "stable" minima of the mean-field free energy given in Eq. (12), assuming that the magnetic unit cells (MUC) are either **(i)** original square-lattice unit cell or **(ii)** 2×2 of the original unit cell. The dimension of the (real-valued) parameter space in each case becomes **(i)** 32(+3) and **(ii)** 128(+12). Starting from a randomly chosen initial point in these multiple dimensional parameter spaces, we perform the Newton-Raphson method, only to reach a certain local minimum of the mean-field free energy E^{mf} (per the magnetic unit cell)

$$E^{\mathrm{mf}} \equiv \frac{J_1}{4}\sum_{\langle jl\rangle\in \mathrm{MUC}}(|\boldsymbol{E}_{jl}|^2+|\boldsymbol{D}_{jl}|^2)+\frac{J_2}{4}\sum_{\langle\langle jl\rangle\rangle\in\mathrm{MUC}}(|\chi_{jl}|^2+|\eta_{jl}|^2) - \frac{1}{16\pi^2}\sum_{\alpha=1}^{\nu}\int\int_{\mathrm{MBZ}} dk_x dk_y |\lambda_\alpha| \quad (25)$$

with **(i)** $\nu=4$ or **(ii)** $\nu=16$. Here, the summation over jl is taken within each magnetic unit cell and λ_α denotes the spinon energy band. We have repeated this procedure from 50 times to 300 times for each parameter point, i.e., $(J_1,J_2)=(\sin\theta,\cos\theta)$ with $0\le\theta\le\frac{\pi}{2}$. In this way, we enumerated various spin-triplet RVB ansatzes.

Throughout this extensive search, we found basically three distinct RVB ansatzes having both spin-triplet link variable on each NN bond and spin-singlet link variable on each NNN bond. All of these three do not break any translational symmetries of the original unit cell, i.e., T_x and T_y.

1. Z_2 Balian-Werthamer state

The first one is a sort of the Balian-Werthamer (BW) state[39] where the $\boldsymbol{d}$ vector on the NN x link is perpendicular to that on the y link

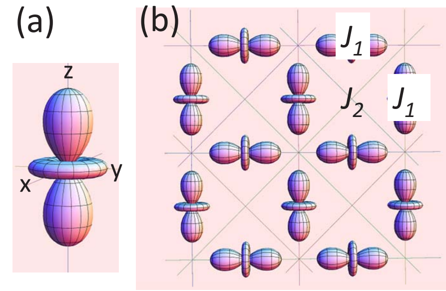

FIG. 1. (Color online) (a) $2z^2-x^2-y^2$ type quadrupole moment formed by two $S=1/2$ spins on each bond. (b) J_1-J_2 model and the configuration of the quadrupole moments on bonds in the Z_2 BW state [see Eqs. (27) and (28)].

$$U^{\mathrm{tri}}_{\langle j,j+\hat{x}\rangle,\mu}=i\delta_{\mu 1}D\sigma_2,\quad U^{\mathrm{tri}}_{\langle j,j+\hat{y}\rangle,\mu}=i\delta_{\mu 2}D\sigma_2,$$

$$U^{\mathrm{sin}}_{\langle j,j+\hat{x}\pm\hat{y}\rangle}=\chi\sigma_3\pm\eta\sigma_1,\quad ia_\nu=0. \quad (26)$$

"D," "χ" and "η" above correspond to the real parts of Eqs. (5) and (8), respectively. This RVB state exhibits the same antiferro-type configuration of quadrupolar moments as the bond-nematic state found in Ref. 5. Namely, the nematic order parameters on NN bonds show

$$Q_{jl,11}=-\frac{2}{3}D^2,\quad Q_{jl,22}=Q_{jl,33}=\frac{1}{3}D^2 \quad (27)$$

for the x direction and

$$Q_{jl,22}=-\frac{2}{3}D^2,\quad Q_{jl,11}=Q_{jl,33}=\frac{1}{3}D^2 \quad (28)$$

for the y direction, where $Q_{jl,\mu\nu}=0$ for $\mu\neq\nu$ (see Fig. 1). While this mean-field ansatz breaks the mirror symmetry P_{xy} which interchanges x link and y link, it is invariant under the following combined symmetry and gauge transformations: G_xT_x, G_yT_y, $G_{P_x}P_x$, $G_{P_y}P_y$, $G_{P'_{xy}}P'_{xy}$, and $G_{\mathcal{T}}\mathcal{T}$. The respective gauge transformations read

$$G_x=G_y=\sigma_0,\quad G_{P_x}=i\sigma_1(-1)^{j_x},\quad G_{P_y}=i\sigma_1(-1)^{j_y},$$

$$G_{P'_{xy}}=i\sigma_2(-1)^{j_y},\quad G_{\mathcal{T}}=(-1)^{i_x+i_y}. \quad (29)$$

Here $\mathcal{T}$ refers to the time-reversal symmetry while P'_{xy} stands for the mirror symmetry P_{xy} accompanied by an appropriate spin rotation about the 3 axis by $\pi/2$.

Provided that $\eta\chi\neq 0$, the ansatz supports two noncollinear $SU(2)$ gauge fluxes

$$U^{\mathrm{tri}}_{\langle j,j+\hat{x}\rangle,1}U^{\mathrm{tri}}_{\langle j+\hat{x},j+\hat{x}+\hat{y}\rangle,2}U^{\mathrm{sin}}_{\langle j+\hat{x}+\hat{y},j\rangle}\propto\chi\sigma_3+\eta\sigma_1, \quad (30)$$

$$U^{\mathrm{tri}}_{\langle j,j+\hat{x}\rangle,1}U^{\mathrm{tri}}_{\langle j+\hat{x},j+\hat{x}-\hat{y}\rangle,2}U^{\mathrm{sin}}_{\langle j+\hat{x}-\hat{y},j\rangle}\propto\chi\sigma_3-\eta\sigma_1. \quad (31)$$

Hence it is protected from any small gauge fluctuation by finite Higgs mass. We call this ansatz as the Z_2 BW state. The spinon's band dispersion λ_α of this Z_2 state is comprised of two doubly degenerate bands, both of which are always separated by a finite-energy gap in the entire Brillouin zone, $[-\pi,\pi]\times[-\pi,\pi]$

$$\lambda_{1,2} \equiv -\lambda_{3,4} \equiv \{A^2(s_x^2+s_y^2)+B^2c_x^2c_y^2+C^2s_x^2s_y^2\}^{1/2} \quad (32)$$

with $(s_\mu,c_\mu)\equiv(\sin k_\mu,\cos k_\mu)$ and $(2A,B,C) \equiv (J_1D,J_2\chi,J_2\eta)$.

2. SU(2) chiral p-wave state

The second ansatz we found is the chiral p-wave [Anderson-Brinkman-Morel] state,[40] in which all the $\boldsymbol{d}$ vectors on the NN bonds are collinear, while the $\boldsymbol{d}$ vector on the x link acquires extra phase i in relative to that on the y link

$$U^{\text{tri}}_{\langle j,j+\hat{x}\rangle,\mu}=i\delta_{\mu 3}D\sigma_2, \quad U^{\text{tri}}_{\langle j,j+\hat{y}\rangle,\mu}=i\delta_{\mu 3}D\sigma_1,$$

$$U^{\text{sin}}_{\langle j,j+\hat{x}\pm\hat{y}\rangle}=\chi\sigma_3, \quad ia_\nu=0. \quad (33)$$

Namely, two D appearing in the first line stand for the real and imaginary part of the $\boldsymbol{d}$ vector, respectively. Because of this relative phase factor, this ansatz has its fermionic band-dispersion fully gapped in the whole momentum space

$$\lambda_{1,2}=-\lambda_{3,4}=\lambda_k=\{A^2(s_x^2+s_y^2)+B^2c_x^2c_y^2\}^{1/2}. \quad (34)$$

In this state, all NN bonds have the same ferronematic order $Q_{jl,33}=-\frac{2}{3}D^2$, $Q_{jl,11}=Q_{jl,22}=\frac{1}{3}D^2$. The IGG of this chiral p-wave state contains the following three continuous gauge symmetries:

$$\{e^{i(-1)^{j_x+j_y}\theta\sigma_3},e^{i(-1)^{j_x}\theta\sigma_1},e^{i(-1)^{j_y}\theta\sigma_2}|\theta\in[0,2\pi)\}. \quad (35)$$

Correspondingly, the low-energy effective theory in the gauge (nonmagnetic) part consists of three maxwell forms around $q=(\pi,\pi)$, $(\pi,0)$, and $(0,\pi)$, respectively. Namely, above continuous gauge symmetries require that the following three types of nonmagnetic deformations constitute the $U(1)$ gauge-invariant effective actions:

$$\{U^{\text{tri}}_{jl,\mu},U^{\text{sin}}_{jl}\}=\{\bar{U}^{\text{tri}}_{jl,\mu},\bar{U}^{\text{sin}}_{jl}\}e^{i(j-l)_\alpha(-1)^{l_x+l_y}a_\alpha(l,\tau)\sigma_3},$$

$$a^3_{j,\tau}=(-1)^{j_x+j_y}a_0(j,\tau), \quad (36)$$

$$\{U^{\text{tri}}_{jl},U^{\text{sin}}_{jl}\}=\{\bar{U}^{\text{tri}}_{jl,\mu},\bar{U}^{\text{sin}}_{jl}\}e^{i(j-l)_\alpha(-1)^{l_x}a_\alpha(l,\tau)\sigma_1},$$

$$a^1_{j,\tau}=(-1)^{j_x}a_0(j,\tau), \quad (37)$$

$$\{U^{\text{tri}}_{jl,\mu},U^{\text{sin}}_{jl}\}=\{\bar{U}^{\text{tri}}_{jl,\mu},\bar{U}^{\text{sin}}_{jl}\}e^{i(j-l)_\alpha(-1)^{l_y}a_\alpha(l,\tau)\sigma_2},$$

$$a^2_{j,\tau}=(-1)^{j_y}a_0(j,\tau). \quad (38)$$

Though these three types of gauge fluctuations are not suppressed by finite Higgs mass, the ansatz itself is still protected by the so-called Chern-Simon mechanism.[33,34,49–53]

To see this, notice that the ansatz [Eq. (33)] breaks all the mirror symmetries P_x, P_y, P_{xy}, and the time-reversal symmetry $\mathcal{T}$. Instead, it is invariant only under these mirror symmetries accompanied by the time-reversal symmetry $G_{P\mathcal{T}}P\mathcal{T}$ or under the spatial-inversion symmetry $G_{R_\pi}R_\pi$. The respective gauge transformations are given by

$$G_{P_y\mathcal{T}}=\sigma_0, \quad G_{\mathcal{R}_\pi}=G_{P_x\mathcal{T}}=(-1)^{j_x+j_y},$$

$$G_{P_{xy}\mathcal{T}}=i(-\sigma_3)^{j_x+j_y}. \quad (39)$$

This magnetic point group clearly allows the spontaneous Hall conductance of the "spinon," like in the chiral spin state.[51–53] In fact, corresponding to the three continuous gauge symmetries given in Eq. (35), we have three conserved "charges," all of which are accompanied by finite quantized transverse conductance $\sigma_{xy}=\frac{2}{2\pi}$. As a result, the effective actions around $q=(\pi,\pi)$, $(0,\pi)$, and $(\pi,0)$ acquire the Chern-Simon term in addition to the maxwell form[33,34,51–53]

$$F_{\text{gauge}}\equiv\int dx^2d\tau\frac{\sigma_{xy}}{2}a_\mu\partial_\nu a_\lambda\epsilon_{\mu\nu\lambda}+(\text{maxwell form}).$$

This Chern-Simon term endows the apparently massless gauge boson with a finite-energy gap.[50]

3. Z_2 collinear state

The third stable ansatz we found is the "collinear" state, where all $\boldsymbol{d}$ vectors are pointing to the same direction

$$U^{\text{tri}}_{\langle j,j+\hat{x}\rangle,\mu}=U^{\text{tri}}_{\langle j,j+\hat{y}\rangle,\mu}=i\delta_{\mu 3}D\sigma_2,$$

$$U^{\text{sin}}_{\langle j,j+\hat{x}\pm\hat{y}\rangle}=\chi\sigma_3\pm\eta\sigma_1, \quad ia^1_{j,\tau}\neq 0 \quad (40)$$

showing ferronematic order $Q_{jl,33}=-\frac{2}{3}D^2$ and $Q_{jl,11}=Q_{jl,22}=\frac{1}{3}D^2$. Although having the same spin-quadrupolar moment as the previous one, this collinear ansatz is a distinct quantum order state from the $SU(2)$ chiral p-wave state. It preserves mirror symmetries as well as the time-reversal symmetry. In fact, one can see that all the discrete symmetries of the original square lattice are recovered, when combined with the following gauge transformations:

$$G_x=G_y=\sigma_0, \quad G_{P_x}=i\sigma_1(-1)^{j_x}, \quad G_{P_y}=i\sigma_1(-1)^{j_y},$$

$$G_{P_{xy}}=1, \quad G_{\mathcal{T}}=(-1)^{j_x+j_y}. \quad (41)$$

Having the noncollinear $SU(2)$ gauge fluxes as in Eqs. (30) and (31), all the gauge fluctuations around this ansatz are suppressed by finite Higgs mass. We hence call this state as Z_2 collinear state.

B. Phase diagram

The mean-field energy for these three ansatzes are plotted in Fig. 2(a) with $(J_1,J_2)\equiv J(\sin\theta,\cos\theta)$. Let us begin with the lowest energy mean-field solution in the well-studied limit, $J_2\gg J_1$. In the strong J_2 limit, our model reduces to the *two decoupled* antiferromagnetic square lattice so that the knowledges of the saddle-point solutions in this limit have been well established.[30,31,33,34,41,55–58] Namely, the π-flux state defined on each square lattice

$$U^{\text{sin}}_{\langle j,j+\hat{x}\pm\hat{y}\rangle}=\chi\sigma_3\pm\eta\sigma_1, \quad U^{\text{tri}}_{\langle j,j+\hat{\nu}\rangle,\mu}=ia^\mu_{j,\tau}=0 \quad (42)$$

with $\chi=\eta$, becomes global minimum, when the MUC is restricted to the original square-lattice unit cell. On the other hand, when the MUC is enlarged up to the 2×2, the global minimum state becomes one of the staggered dimer states introduced on each decoupled square lattice, e.g.,

$$U^{\text{sin}}_{\langle j,j+\hat{x}+\hat{y}\rangle} = U^{\text{sin}}_{\langle j+\hat{x},j+\hat{y}\rangle} = \chi\sigma_3, \quad U^{\text{tri}}_{\langle j,j+\hat{\nu}\rangle,\mu} = ia^{\mu}_{j,\tau} = 0,$$

$$U^{\text{sin}}_{\langle j+\hat{x},j+2\hat{x}+\hat{y}\rangle} = U^{\text{sin}}_{\langle j+2\hat{x},j+\hat{x}+\hat{y}\rangle} = 0,$$

$$U^{\text{sin}}_{\langle j+\hat{y},j+\hat{x}+2\hat{y}\rangle} = U^{\text{sin}}_{\langle j+\hat{x}+\hat{y},j+2\hat{y}\rangle} = 0,$$

$$U^{\text{sin}}_{\langle j+\hat{x}+\hat{y},j+2\hat{x}+2\hat{y}\rangle} = U^{\text{sin}}_{\langle j+2\hat{x}+\hat{y},j+\hat{x}+2\hat{y}\rangle} = 0. \tag{43}$$

However, using the variational Monte Carlo calculations, Gros and his co-workers[41] have demonstrated that, when projected onto the original (spin) Hilbert space, the π-flux state eventually wins over this isolated dimer state. In fact, it is well established[58] that the projected π-flux state gives the second best variational energy in the strong J_2 limit (the best variational estimate is obtained from the Neel order state[56]).

When increasing the NN ferromagnetic interaction J_1, a finite spin-triplet ansatz continuously develops on the top of this π-flux state, while simultaneously the parameters η start to deviate from χ, i.e., $\eta \neq \chi$. This leads to either Z_2 BW state or Z_2 collinear state for $\theta_{c1} \equiv 0.66 < \theta$. Thus, the transitions from the π-flux state to these two Z_2 states are both the second order at the mean-field level. Energetically speaking, the Z_2 BW state gives a slightly lower mean-field energy than that of the Z_2 collinear state.

Notice also that these two Z_2 states are clearly preemptted by the staggered dimer state, Eq. (43), at the mean-field level [see Fig. 2(a)]. Observing the situation in the strong J_2 limit, however, one can naturally expect that, when projected onto the physical (spin) Hilbert space, both Z_2 states would win over this isolated dimer state in the case of a finite J_1. Namely, since our Z_2 states are constructed based on the decoupled π-flux states [compare Eqs. (26) and (40) with Eq. (42)], they would certainly acquire substantial resonance energies in the same way as the π-flux state does. On the other hand, being factorisable, any isolated dimer state cannot gain such resonance energies, irrespective of finite ferromagnetic exchange interactions. Moreover, Fig. 2(a) indicates that the Z_2 BW asatz is quite energetically tunable in the presence of the ferromagnetic exchange interaction. Thus, we presume that the Z_2 BW state finally dominates in this intermediate coupling region, $\theta_{c1} \equiv 0.66 < \theta$.

When $\theta_{c2} \equiv 0.76 < \theta$, this Z_2 BW state reduces to the $U(1)$ state having no finite η. Namely, with $\eta=0$, two $SU(2)$ gauge fluxes given in Eqs. (30) and (31) become collinear with each other. Simultaneously, this $U(1)$ BW state becomes energetically degenerate with the $SU(2)$ chiral p-wave state. Namely, both of them have precisely the same mean-field band dispersions $\pm\lambda_k$ [compare Eq. (34) with Eq. (32) having $\eta=0$].

This $U(1)$ BW state is destroyed by the infinitesimally small gauge fluctuation. Namely, in the absence of finite η, the nonmagnetic deformations defined in Eq. (19) constitute the following maxwell form around $q=(\pi,\pi)$:

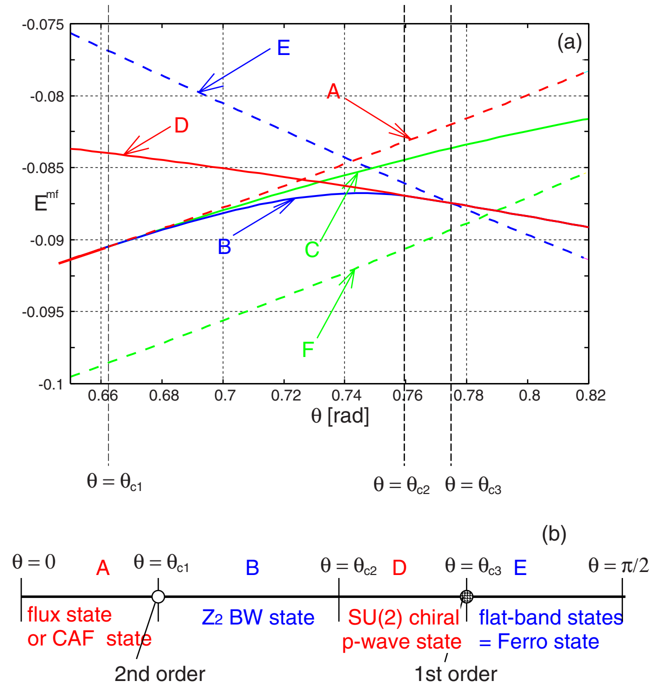

FIG. 2. (Color online) (a) Mean-field energies (per site) of the various ansatzes in the $S=1/2$ square-lattice ferromagnetic J_1-J_2 model. Note that $(J_1,J_2)\equiv|J|(\sin\theta,\cos\theta)$, where the energy unit is taken to be $|J|$. The blue line (labeled as B) is for the BW state, which is the Z_2 state for $\theta_{c1}<\theta<\theta_{c2}$ and which reduces to the $U(1)$ state for $\theta_{c2}<\theta$. The green line (labeled as C) is for the Z_2 collinear state while the red line (labeled as D) stands for the $SU(2)$ chiral p-wave state. The red-dotted line (labeled as A) is the doubled π-flux state, where both the A sublattice and the B sublattice support π-flux states, respectively [see Eq. (42)]. The Blue-dotted line (labeled as E) is for a set of flat-band states ($E^{\text{mf}}_{\text{flat}}=-\frac{1}{8}|J|\sin\theta$), all of which give the same best mean-field energy for $\theta_{c3}<\theta$. The green-dotted line (labeled as F) is for the staggered dimer state ($E^{\text{mf}}_{\text{dimer}}=-\frac{1}{8}|J|\cos\theta$), where both the A sublattice and the B sublattice support staggered dimer states, respectively [see Eq. (43) for its example]. These isolated dimer states are known to be overcome energetically by the doubled π-flux state (Ref. 41), when they are projected onto the physical Hilbert space. Since the Z_2 BW state is composed on the top of the π-flux state, this staggered dimer state is also expected to be overcome by the *projected* Z_2 BW state. (b) Expected mean-field phase diagram in the intermediate coupling region. The transition at θ_{c1} is the second order since the magnetic space group of the Z_2 BW state belongs to that of the π-flux state. On the other hand, the transition at θ_{c3} is the first order at the mean-field level, which one can see directly from the figure (a).

$$F_{\text{gauge}} = \int_0^{\beta} d\tau \int d^2x \left\{ ue^2 + \frac{1}{2}Kb^2 \right\} + \cdots,$$

where e_α $(\alpha=1,2)$ and b are defined, from Eqs. (19) and (20), as $e_\alpha(j,\tau)\equiv(-1)^{j_x+j_y}(\partial_\tau a_\alpha - \partial_\alpha a_0)$ and $b(j,\tau)\equiv(-1)^{j_x+j_y}(\partial_2 a_1 - \partial_1 a_2)$. Since the fermionic excitations are fully gapped even without η [see Eq. (32)], this maxwell form is free from any dissipation effect,[59] e.g.,

$$u = \int\int_{[-\pi,\pi]^2} d^2k \frac{A^4 s_y^2 c_x^2 + A^2B^2c_y^2(1+s_x^2s_y^2)}{16\pi^2\lambda_k^5} > 0.$$

Having the time-reversal symmetry [see Eq. (29)], the massless gauge fluctuation is not suppressed by the Chern-Simon term either.[60] Consequently, infinitesimally small fluctuations of this type of gauge fields lead the $U(1)$ BW state into a confining phase having no gapped free spinon in its excitation. More specifically, those space-time instantons (monopoles) which are allowed by the corresponding compact QED action, $\int d\tau \int d^2x\{u\boldsymbol{e}^2 - K\cos(\epsilon_{\alpha\beta}\Delta_\alpha a_\beta)\}$, proliferate in the 2+1 dimensional space,[38] lowering a certain magnetic symmetries enumerated in Eq. (29).[61] To capture the resulting magnetic space group of the confining phase, one generally needs to identify the quantum number carried by this monopole creation field.[62,63]

For $\theta_{c3} \equiv 0.775 < \theta$, these two degenerate ansatzes—$SU(2)$ chiral p-wave state and $U(1)$ BW state—are further overcome (energetically) by another ansatz, which we dubbed as the "flat-band" states

$$U_{jl}^{\text{sin}} = 0, \quad U_{jl}^{\text{tri}} \neq 0, \quad ia_{j,\tau}^{\nu} \neq 0. \tag{44}$$

These flat-band states do not have any finite singlet ansatzes anymore and keep on giving the lowest mean-field energy ($E_{\text{flat}}^{\text{mf}} = -\frac{1}{8}J_1$) for the remaining ferromagnetic side, $\theta_{c3} < \theta < \frac{\pi}{2}$. However, these flat-band states do not necessarily refer to a specific configuration of the spin-triplet ansatzes. Instead, they refer to a *group* of the states all of which give precisely the same mean-field energy. For example, these flat-band states include the following parameterization of the spin-triplet ansatz:

$$[\boldsymbol{E}''_{\langle j,j+\hat{x}\rangle} \quad \boldsymbol{D}''_{\langle j,j+\hat{x}\rangle} \quad \boldsymbol{D}'_{\langle j,j+\hat{x}\rangle}] = \alpha \boldsymbol{n}_1 \cdot \boldsymbol{m}_1^T,$$

$$[\boldsymbol{E}''_{\langle j,j+\hat{y}\rangle} \quad \boldsymbol{D}''_{\langle j,j+\hat{y}\rangle} \quad \boldsymbol{D}'_{\langle j,j+\hat{y}\rangle}] = \beta \boldsymbol{n}_1 \cdot \boldsymbol{m}_2^T,$$

$$\boldsymbol{E}'_{\langle j,j+\hat{x}\rangle} = \alpha \boldsymbol{n}_2, \quad \boldsymbol{E}'_{\langle j,j+\hat{y}\rangle} = \beta \boldsymbol{n}_3,$$

$$\boldsymbol{n}_1^T \cdot \boldsymbol{n}_2 = \boldsymbol{n}_2^T \cdot \boldsymbol{n}_3 = \boldsymbol{n}_3^T \cdot \boldsymbol{n}_1 = \boldsymbol{m}_1^T \cdot \boldsymbol{m}_2 = 0, \tag{45}$$

where $\alpha^2 + \beta^2 = \frac{1}{4}$, and $\boldsymbol{n}_j$ and $\boldsymbol{m}_j$ can be arbitrary unit vectors that observe Eq. (45). Here $E'_{jl,\mu}$ and $D'_{jl,\mu}$ stand for the real part of $E_{jl,\mu}$ and $D_{jl,\mu}$, respectively, while $E''_{jl,\mu}$ and $D''_{jl,\mu}$ are their respective imaginary parts. Thus, only the first one is parity even $E'_{jl,\mu} = E'_{lj,\mu}$ while the others are odd $E''_{jl,\mu} = -E''_{lj,\mu}$ and $D_{jl,\mu} = -D_{lj,\mu}$. Bearing these in mind, one can easily see that this mean-field ansatz always gives the two doubly degenerate spinon bands, which are totally *flat* in the entire Brillouin zone

$$\lambda_{1,2} = -\lambda_{3,4} = \frac{J_1}{4}.$$

Because of this feature, all the spin-triplet ansatzes parameterized by Eq. (45) give the same mean-field energy (per site) $E_{\text{flat}}^{\text{mf}} = -\frac{1}{8}J_1$. The emergence of these "huge" numbers of flat-band states in the strong J_1 limit reflects the fact that the ground-state order parameter of any Heisenberg ferromagnet (total spin moment) and the corresponding spin Hamiltonian are *simultaneously diagonalizable*. When projected onto the physical (spin) Hilbert space, we expect that these flat-band states reduce to a fully polarized state (ferromagnetic state).

Observing Fig. 2, please notice that our Z_2 BW phase appears in larger J_2 region in comparison with the previous ED studies. Namely, Fig. 2 indicates that its phase boundaries are given by $J_1 : J_2 = 1 : 1.29$ at $\theta = \theta_{c1}$ and $J_1 : J_2 = 1 : 1.05$ at $\theta = \theta_{c2}$ while d-wave bond-nematic order phase was found in $0.4 \lesssim J_2/J_1 \lesssim 0.6$ in the previous finite-size studies.[5] This discrepancy simply stems from the so-called "factor 3" difference, often encountered between the Hartree-Fock (HF) spin-singlet ansatz and the HF spin-triplet ansatz. If one employed a more numerics-oriented formulation[64] $\frac{J_1}{4}$ appearing in Eq. (12) is replaced by $\frac{J_1}{8}$ while $\frac{J_2}{4}$ is replaced by $\frac{3J_2}{8}$. Consequently, we have $J_1 : J_2 = 1 : 0.43$ ($\theta = \theta_{c1}$) and $J_1 : J_2 = 1 : 0.36$ ($\theta = \theta_{c2}$), which would be relatively comparable with the previous ED result. More quantitative comparison, however, requires the variational Monte Carlo studies based on these spin-triplet ansatzes.

In summary, we have argued that three spin-triplet RVB ansatzes—Z_2 and $U(1)$ BW states and $SU(2)$ chiral p-wave state—become the lowest mean-field states in the intermediate coupling region, $J_1 \simeq 2J_2$ [see Fig. 2(b)]. Among them, both the Z_2 BW state and the $SU(2)$ chiral p-wave state are stable against any (infinitesimally) small gauge fluctuation while in the $U(1)$ BW state the effect of gauge fluctuation is crucial making spinons confined. Using Eq. (6), one can easily see that the BW states show the d-wave bond-type spin-quadrupolar order precisely as in Eq. (3).

C. Magnetic excitations in the BW states

Here we briefly discuss magnetic excitations in the Z_2 BW state. The "low-energy" excitation around the Z_2 BW state is composed of three parts; **(i)** gapped nonmagnetic excitations (gauge bosons), **(ii)** gapless magnetic excitations (Goldstone bosons), and **(iii)** gapped fermionic (Ψ field) individual excitations. The gapped gauge boson plays only a subdominant role in the spin-structure factor while the latter two contribute significantly to magnetic excitations. Up to the Hartree-Fock level, one can easily see that the gapped fermionic excitation constitutes the continuum spectrum above $\omega > \max(J_1|D|, J_2|\chi|)$. When one further takes into account the random-phase approximation terms,[65,66] the gapless bosonic dispersions emerge below this spinon continuum, whose low-energy limit can be described by the matrix-formed nonlinear σ model

$$F_{\text{magnetic}} = \sum_{\mu=\tau,x,y} \text{Tr}[\hat{\Lambda}_\mu \partial_\mu \hat{R}^{-1} \partial_\mu \hat{R}]. \tag{46}$$

Namely, the 3×3 matrix $\hat{R}$ is nothing but the spatiotemporally varying rotational matrix of the director vector used in Eq. (16). The symmetry argument[67] dictates that the diagonal matrices $\hat{\Lambda}_\mu$ generally take the following form

$$\{\hat{\Lambda}_\tau,\hat{\Lambda}_x,\hat{\Lambda}_y\}\equiv\left\{\begin{bmatrix} c_0 & & \\ & c_2 & \\ & & c_2\end{bmatrix},\begin{bmatrix} c_1 & & \\ & c_3 & \\ & & c_4\end{bmatrix},\begin{bmatrix} c_1 & & \\ & c_4 & \\ & & c_3\end{bmatrix}\right\}, \tag{47}$$

where the director coplanar plane was taken to be the 2–3 plane. In terms of the semiclassical (gradient) expansion, one can directly calculate their respective coupling constants:

$$c_0\equiv 0,\quad c_1\equiv\int\int_{[-\pi,\pi]^2} d^2k\frac{A^4 s_y^2 c_x^2}{64\pi^2\lambda_k^3},$$

$$c_2\equiv\int\int_{[-\pi,\pi]^2} d^2k\frac{A^2(s_x^2+s_y^2)}{64\pi^2\lambda_k^3},$$

$$c_3\equiv\int\int_{[-\pi,\pi]^2} d^2k\left\{\frac{\mathcal{J}_1 A^2 s_x^2}{8\pi^2}-\frac{A^4 s_y^2 c_x^2}{64\pi^2\lambda_k^3}\right\},$$

$$c_4\equiv\int\int_{[-\pi,\pi]^2} d^2k\left\{\frac{\mathcal{J}_1 A^2 s_y^2}{8\pi^2}-\frac{A^4 s_y^2 c_x^2}{64\pi^2\lambda_k^3}\right\}$$

with $\mathcal{J}_1\equiv 4^{-1}\lambda_k^{-1}\{(\partial_{k_x}\boldsymbol{n}^T)(\partial_{k_x}\boldsymbol{n})+2^{-1}\lambda_k^{-1}\partial_{k_x}^2\lambda_k\}$ and $\boldsymbol{n}\equiv 2^{-1}\lambda_k^{-1}(2As_x,2As_y,Bc_xc_y,Cs_xs_y)$. In addition to these massless excitations, we could also have several gapped ("optical") magnetic modes, provided that they are not damped by the spinon individual excitations.[65,66] One might also expect a certain characteristic behavior of the spectral weight themselves. In fact, Tsunetsugu *et al.*[18] and Lauchli *et al.*[19] demonstrated that the spin-structure factor in the site-nematic ordered state exhibits the *vanishing* spectral intensities of the Goldstone modes around the Γ point.

IV. SUMMARY AND OPEN ISSUES

In this paper, we have introduced the spin-triplet slave-boson formulation as a mean-field theory for the bond-type spin nematic state, which was described as the spin-triplet RVB state. Namely, the $\boldsymbol{d}$ vectors of the spin-triplet RVB ansatz constitute the quadrupolar order while the combination of the spin-triplet and singlet link variables on the same link leads to the vector chiral order.

When applied to the $S=\frac{1}{2}$ square-lattice frustrated ferromagnetic Heisenberg model, our spin-triplet slave-boson analysis gives two nontrivial stable spin-triplet RVB ansatzes in the intermediate coupling region around $J_1:J_2\simeq 1:0.4$. One is the Z_2 BW state stabilized by the Anderson-Higgs mechanism while the other is the $SU(2)$ chiral p-wave state protected by the Chern-Simon mechanism. Our slave-boson analysis also found an unstable $U(1)$ BW state as a mean-field solution, which possibly gives a route to the realization of spinon-confined quadrupolar ordered states with a certain symmetry reduction. The projective symmetry group of the Z_2 BW state as well as the $U(1)$ BW state is consistent with the magnetic space group of the d-wave bond-type spin nematic state discussed in Ref. 5. Both of them exhibit the antiferro-type configuration of the bond quadrupolar moment shown in Fig. 1.

Contrary to a naive expectation, our BW state is classified into a "weak topological (ordinary) insulator" instead of the "strong topological insulator" defined in the recent literatures.[42–46] Physically speaking, such a "weak topological insulator (WTI)" is accompanied either by no spinon edge states at all or by even numbers of the helical edge states. To see that it is indeed a "WTI," one can first deform this Z_2 ansatz into the $U(1)$ ansatz ($\eta\to 0$). Since the fermionic dispersion remains gapped, the $\mathcal{Z}_2$ topological index associated with the filled spinon band[42,44] is also unchanged. After reaching the simpler $U(1)$ ansatz, let us then utilize the Fermi-surface argument recently introduced by Sato.[68] His argument relates the $\mathcal{Z}_2$ topological index in the superconducting state ($D\neq 0$) with the Fermi-surface topology in the corresponding "normal" state ($D=0$). That is, if a Fermi surface in the normal state surrounds odd/even numbers of the time-reversal invariant momentum points, the BW state constructed on top of this normal state is accompanied by nontrivial/trivial $\mathcal{Z}_2$ topological index. Since our normal state is composed of *two* decoupled u-RVB states at $\eta=0$, the resulting Fermi surface clearly surrounds *two* time-reversal symmetric k points, i.e., (0,0) and (π,π). Thus, our Z_2 BW state should be classified into the "WTI" ($\mathcal{Z}_2$ even class).

In the followings, we will enumerate several open issues and possible extensions of the current work. The most immediate open issue is to identify the magnetic space group of the confining phase proximate to the $U(1)$ BW state based on the monopole field studies.[62,63] Namely, such an analysis gives several complementary informations to the direct ED studies of the original spin model.[5]

The fate of the $SU(2)$ chiral p-wave state observed at $\theta_{c2}<\theta<\theta_{c3}$ is not so clear either, although we have argued its stability against any (infinitesimally) small gauge fluctuation. Namely, previous exact diagonalization studies of the $SU(2)$ spin model did not find any $\mathcal{T}$-symmetry-breaking ferronematic states between the d-wave bond-nematic state and ferromagnetic state. In fact, it is also possible that, when projected onto the real (spin) Hilbert space, the strong gauge fluctuation could wipe out this time-reversal breaking ansatz.

Though we have mainly discussed the quadrupolar order in this paper, our formulation can also describe vector chiral order having no quadrupolar moment,[4] i.e., $P_{jl,\lambda}\neq 0$ and $Q_{jl,\mu\nu}=0$. In fact, such vector chiral order state was observed in the spin one half frustrated Heisenberg model having the ring-exchange coupling.[4] When applying the current spin-triplet slave-boson formulation onto these quantum-spin systems, one could use the following mean-field parameterization:

$$[\boldsymbol{E}''_{jl}\quad \boldsymbol{D}''_{jl}\quad \boldsymbol{D}'_{jl}]=[\boldsymbol{n}_1\quad \boldsymbol{n}_2\quad \boldsymbol{n}_3],$$

$$[\chi'_{jl}\quad -\eta'_{jl}\quad \eta''_{jl}]=[\gamma_1\quad \gamma_2\quad \gamma_3],$$

where $\{\boldsymbol{n}_1,\boldsymbol{n}_2,\boldsymbol{n}_3\}$ are the normalized unit vectors orthogonal to one another. Namely, such an ansatz gives a finite vector chirality, $\boldsymbol{P}_{jl}\equiv 2i\Sigma_{\alpha=1}^3\gamma_\alpha\boldsymbol{n}_\alpha$ without any quadrupolar moments. We generally have three alternative ways to parameterize this vector chiral order

$$[\boldsymbol{E}'_{jl} \quad \boldsymbol{D}''_{jl} \quad \boldsymbol{D}'_{jl}] = [\boldsymbol{n}_1 \quad \boldsymbol{n}_2 \quad \boldsymbol{n}_3],$$

$$[\chi''_{jl} \quad -\eta'_{jl} \quad \eta''_{jl}] = [\gamma_1 \quad \gamma_2 \quad \gamma_3]$$

or

$$[\boldsymbol{E}''_{jl} \quad \boldsymbol{E}'_{jl} \quad \boldsymbol{D}'_{jl}] = [\boldsymbol{n}_1 \quad \boldsymbol{n}_2 \quad \boldsymbol{n}_3],$$

$$[\chi'_{jl} \quad \chi''_{jl} \quad \eta''_{jl}] = [\gamma_1 \quad \gamma_2 \quad \gamma_3]$$

or

$$[\boldsymbol{E}''_{jl} \quad \boldsymbol{D}''_{jl} \quad \boldsymbol{E}'_{jl}] = [\boldsymbol{n}_1 \quad \boldsymbol{n}_2 \quad \boldsymbol{n}_3],$$

$$[\chi'_{jl} \quad -\eta'_{jl} \quad \chi''_{jl}] = [\gamma_1 \quad \gamma_2 \quad \gamma_3].$$

ACKNOWLEDGMENTS

We acknowledge Takuma Ohashi, Sung-Sik Lee, Hosho Katsura, Naoto Nagaosa, Yong Baek Kim, Leon Balents, Seiji Yunoki, Masao Ogata, Nic Shannon, Philippe Sindzingre, Keisuke Totsuka, and Akira Furusaki for helpful discussions and encouragement. We are especially grateful to Sung-Sik Lee for clarifying the symmetry reduction induced by the monopole proliferations, to Keisuke Totsuka for clarifying the matrix formed NLσM, to Seiji Yunoki for his advice on the efficient coding of the Newton-Raphson method. We are also grateful to Takuma Ohashi for his collaboration in Schwinger boson formulation in the early stage of this work. R.S. was supported by the Institute of Physical and Chemical Research (RIKEN) and T.M. was supported by Grants-in-Aid for Scientific Research from the Ministry of Education, Culture, Sports, Science and Technology (MEXT) of Japan (Grants No. 17071011 and No. 20046016). Part of this work was done during the international workshop "Topological Aspects of Solid State Physics (TASSP)," which was supported by the Institute for Solid State Physics (ISSP), University of Tokyo, Yukawa Institute, Kyoto University, and I2CAM (with U.S. NSF I2CAM International Materials Institute Award under Grant No. DMR-0645461).

APPENDIX: LARGE N FRUSTRATED FERROMAGNETIC MODEL

The mean-field analysis described in this paper becomes exact in the large N limit of the following action:

$$Z = \int dU^{\text{sin}} dU^{\text{tri}} d\vec{a}_\tau d\Psi^{a\dagger} d\Psi^a \exp\left[-\int_0^\beta d\tau \mathcal{L}\right], \quad \text{(A1)}$$

$$\mathcal{L} = \frac{1}{2}\sum_j \text{Tr}\left[\Psi_j^{a\dagger}\left(\partial_\tau \sigma_0 + \sum_{\nu=1}^{3} i a^\nu_{j,\tau}\sigma_\nu\right)\Psi_j^a\right] - \frac{J_1}{4}\sum_{\langle jl\rangle}\{N(-|\boldsymbol{E}_{jl}|^2 - |\boldsymbol{D}_{jl}|^2) + \text{Tr}[\Psi_j^{a\dagger} U^{\text{tri}}_{jl,\mu}\Psi_l^a \sigma_\mu^T]\} - \frac{J_2}{4}\sum_{\langle\langle jl\rangle\rangle}\{N(-|\chi_{jl}|^2 - |\eta_{jl}|^2) + \text{Tr}[\Psi_j^{a\dagger} U^{\text{sin}}_{jl}\Psi_l^a]\}, \quad \text{(A2)}$$

where the summations with respect to the fermion's species index a $(=1,\ldots,N)$ were made implicit. The integration over the auxiliary fields leads the following large N spin Hamiltonian for frustrated ferromagnets:

$$\mathcal{H}_N \equiv -\frac{J_1}{N}\sum_{\langle jl\rangle}\{\boldsymbol{S}_j^{ab}\cdot\boldsymbol{S}_l^{ba} + \psi_j^{ab}\psi_l^{ba}\} + \frac{J_2}{N}\sum_{\langle\langle jl\rangle\rangle}\boldsymbol{S}_j^{ab}\cdot\boldsymbol{S}_l^{ba}. \quad \text{(A3)}$$

Note that, in addition to the usual $SP(2N)$ spin operators,[33] we have the density operator which is asymmetric in the fermion's species index

$$\psi^{ab} \equiv \frac{i}{2}(f_\alpha^{a\dagger}f_\alpha^b - f_\alpha^{b\dagger}f_\alpha^a), \quad S^{ab3} \equiv \frac{1}{2}(f_\uparrow^{a\dagger}f_\uparrow^b - f_\downarrow^{b\dagger}f_\downarrow^a),$$

$$S^{ab+} \equiv \frac{1}{2}(f_\uparrow^{a\dagger}f_\downarrow^b + f_\uparrow^{b\dagger}f_\downarrow^a), \quad S^{ab-} \equiv \{S^{ab+}\}^\dagger. \quad \text{(A4)}$$

The Hilbert space of this generalized spin Hamiltonian is defined as the $SU(2)$-gauge-invariant subspace of the fermionic Hilbert space.[33] That is, any fermion wave function which respects the following local constraints is an element of our Hilbert space:

$$\left\{\sum_{a=1}^{N} f_{j\alpha}^{a\dagger}[\hat{\sigma}_\mu]_{\alpha\beta}f_{j\beta}^a\right\}|\text{phy}\rangle \equiv 0, \quad \forall\, j,\mu = 1,2,3.$$

The density and spin operators defined in Eq. (A4) in fact act within this physical Hilbert space. Moreover, they observe the following commutation relations:

$$[S^{ab3}, S^{cd3}] = \frac{1}{2}(\delta^{bc}S^{ad3} - \delta^{ad}S^{cb3}),$$

$$[S^{ab3}, S^{cd+}] = \frac{1}{2}(\delta^{bc}S^{ad+} + \delta^{bd}S^{ac+}),$$

$$[S^{ab3}, S^{cd-}] = -\frac{1}{2}(\delta^{ad}S^{bc-} + \delta^{ac}S^{bd-}),$$

$$[S^{ab+}, S^{cd+}] = 0, \quad [S^{ab-}, S^{cd-}] = 0,$$

$$[S^{ab+}, S^{cd-}] = \frac{1}{2}(\delta^{ac}S^{bd3} + \delta^{ad}S^{bc3} + \delta^{bc}S^{ad3} + \delta^{bd}S^{ac3}),$$

$$[\psi^{ab}, S^{cd\pm}] = \frac{i}{2}(\delta^{bc}S^{ad\pm} - \delta^{ac}S^{bd\pm} - \delta^{ad}S^{cb\pm} + \delta^{bd}S^{ca\pm}),$$

$$[\psi^{ab}, S^{cd3}] = \frac{i}{2}(\delta^{bc}S^{ad3} - \delta^{ac}S^{bd3} - \delta^{ad}S^{cb3} + \delta^{bd}S^{ca3}),$$

$$[\psi^{ab},\psi^{cd}] = \frac{i}{2}(\delta^{bc}\psi^{ad} - \delta^{ac}\psi^{bd} - \delta^{ad}\psi^{cb} + \delta^{bd}\psi^{ca}). \tag{A5}$$

Using them, one can argue that the generalized spin Hamiltonian given in Eq. (A3) is invariant under those continuous symmetries which are generated by

$$\psi^{ab}_{\mathrm{tot}},\ \sum_{a=1}^{N} S^{aa3}_{\mathrm{tot}},\ \sum_{a=1}^{N} S^{aa1}_{\mathrm{tot}},\ \sum_{a=1}^{N} S^{aa2}_{\mathrm{tot}}.$$

When $N=1$, ψ^{ab} disappears by itself and Eq. (A3) in combination with Eq. (A5) reduces to the $SU(2)$ Heisenberg spin model defined in Eq. (2).

[1] A. F. Andreev and A. Grishchuk, Sov. Phys. JETP **60**, 267 (1984).
[2] A. V. Chubukov, Phys. Rev. B **44**, 4693 (1991).
[3] P. Chandra and P. Coleman, Phys. Rev. Lett. **66**, 100 (1991).
[4] A. Läuchli, J. C. Domenge, C. Lhuillier, P. Sindzingre, and M. Troyer, Phys. Rev. Lett. **95**, 137206 (2005).
[5] N. Shannon, T. Momoi, and P. Sindzingre, Phys. Rev. Lett. **96**, 027213 (2006).
[6] T. Vekua, A. Honecker, H.-J. Mikeska, and F. Heidrich-Meisner, Phys. Rev. B **76**, 174420 (2007).
[7] T. Hikihara, L. Kecke, T. Momoi, and A. Furusaki, Phys. Rev. B **78**, 144404 (2008).
[8] T. Momoi and N. Shannon, Prog. Theor. Phys. Suppl. **159**, 72 (2005).
[9] T. Momoi, P. Sindzingre, and N. Shannon, Phys. Rev. Lett. **97**, 257204 (2006).
[10] T. Momoi, K. Kubo, and P. Sindzingre (unpublished).
[11] H. H. Chen and P. M. Levy, Phys. Rev. Lett. **27**, 1383 (1971).
[12] P. M. Chaikin and T. C. Lubensky, *Principles of Condensed Matter Physics* (Cambridge University Press, Cambridge, 1995).
[13] V. M. Matveev, Sov. Phys. JETP **38**, 813 (1974).
[14] A. Chubukov, J. Phys.: Condens. Matter **2**, 1593 (1990).
[15] N. Papanicolaou, Nucl. Phys. B **240**, 281 (1984).
[16] A. Joshi, M. Ma, F. Mila, D. N. Shi, and F. C. Zhang, Phys. Rev. B **60**, 6584 (1999).
[17] K. Harada and N. Kawashima, Phys. Rev. B **65**, 052403 (2002).
[18] H. Tsunetsugu and M. Arikawa, J. Phys. Soc. Jpn. **75**, 083701 (2006).
[19] A. Läuchli, F. Mila, and K. Penc, Phys. Rev. Lett. **97**, 087205 (2006).
[20] B. A. Ivanov and A. K. Kolezhuk, Phys. Rev. B **68**, 052401 (2003).
[21] H. Kageyama, T. Kitano, N. Oba, M. Nishi, S. Nagai, K. Hirota, L. Viciu, J. B. Wiley, J. Yasuda, Y. Baba, Y. Ajiro, and K. Yoshimura, J. Phys. Soc. Jpn. **74**, 1702 (2005).
[22] N. Oba, H. Kageyama, T. Kitano, J. Yasuda, Y. Baba, M. Nishi, K. Hirota, Y. Narumi, M. Hagiwara, K. Kindo, T. Saito, Y. Ajiro, and K. Yoshimura, J. Phys. Soc. Jpn. **75**, 113601 (2006).
[23] Y. Tsujimoto, Y. Baba, N. Oba, H. Kageyama, T. Fukui, Y. Narumi, K. Kindo, T. Saito, M. Takano, Y. Ajiro, and K. Yoshimura, J. Phys. Soc. Jpn. **76**, 063711 (2007).
[24] S.-L. Drechsler, N. Tristan, R. Klingeler, B. Büchner, J. Richter, J. Málek, O. Volkova, A. Vasiliev, M. Schmitt, A. Ormeci, C. Loison, W. Schnelle, and H. Rosner, J. Phys.: Condens. Matter **19**, 145230 (2007), and references therein.
[25] S.-L. Drechsler, O. Volkova, A. N. Vasiliev, N. Tristan, J. Richter, M. Schmitt, H. Rosner, J. Málek, R. Klingeler, A. A. Zvyagin, and B. Büchner, Phys. Rev. Lett. **98**, 077202 (2007).
[26] E. E. Kaul, H. Rosner, N. Shannon, R. V. Shpanchenko, and C. Geibel, J. Magn. Magn. Mater. **272-276**, 922 (2004); S. Kini, E. E. Kaul, and C. Geibel, J. Phys.: Condens. Matter **18**, 1303 (2006).
[27] R. Nath, A. A. Tsirlin, E. E. Kaul, M. Baenitz, N. Buttgen, C. Geibel, and H. Rosner, Phys. Rev. B **78**, 024418 (2008).
[28] E. Collin, S. Triqueneaux, R. Harakaly, M. Roger, C. Bäuerle, Yu. M. Bunkov, and H. Godfrin, Phys. Rev. Lett. **86**, 2447 (2001).
[29] J. B. Goodenough, J. Phys. Chem. Solids **6**, 287 (1958); J. Kanamori, *ibid.* **10**, 87 (1959).
[30] P. W. Anderson, Science **235**, 1196 (1987); G. Baskaran, Z. Zou, and P. W. Anderson, Solid State Commun. **63**, 973 (1987); G. Baskaran and P. W. Anderson, Phys. Rev. B **37**, 580 (1988); E. Dagotto, E. Fradkin, and A. Moreo, *ibid.* **38**, 2926 (1988).
[31] G. Kotliar, Phys. Rev. B **37**, 3664 (1988); J. B. Marston and I. Affleck, *ibid.* **39**, 11538 (1989); I. Affleck and J. B. Marston, *ibid.* **37**, 3774 (1988).
[32] Y. Suzumura, Y. Hasegawa, and H. Fukuyama, J. Phys. Soc. Jpn. **57**, 401 (1988); **57**, 2768 (1988).
[33] X. G. Wen, *Quantm Field Theory of Many-Body Systems*, (Oxford University Press, Oxford, 2004); X. G. Wen, Phys. Rev. B **65**, 165113 (2002).
[34] E. Fradkin, *Field Theories of Condensed Matter Systems* (Addison-Wesley, London, 1991).
[35] M. Sigrist and K. Ueda, Rev. Mod. Phys. **63**, 239 (1991).
[36] Some part of this expression was already pointed out by Y. Ran and X. G. Wen, arXiv:cond-mat/0609620 (unpublished).
[37] I. Affleck, Z. Zou, T. Hsu, and P. W. Anderson, Phys. Rev. B **38**, 745 (1988).
[38] A. M. Polyakov, Nucl. Phys. B **120**, 429 (1977).
[39] R. Balian and N. R. Werthamer, Phys. Rev. **131**, 1553 (1963).
[40] P. W. Anderson and P. Morel, Phys. Rev. **123**, 1911 (1961); P. W. Anderson and W. F. Brinkman, Phys. Rev. Lett. **30**, 1108 (1973).
[41] C. Gros, Phys. Rev. B **38**, 931 (1988); F. C. Zhang, C. Gros, T. M. Rice, and H. Shiba, Supercond. Sci. Technol. **1**, 36 (1988).
[42] C. L. Kane and E. J. Mele, Phys. Rev. Lett. **95**, 146802 (2005).
[43] C. L. Kane and E. J. Mele, Phys. Rev. Lett. **95**, 226801 (2005); B. A. Bernevig and S.-C. Zhang, *ibid.* **96**, 106802 (2006).
[44] L. Fu and C. L. Kane, Phys. Rev. B **74**, 195312 (2006).
[45] X. L. Qi, T. L. Hughes, and S. C. Zhang, Phys. Rev. B **78**, 195424 (2008).
[46] A. P. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig, Phys. Rev. B **78**, 195125 (2008).
[47] Y. Nambu, Phys. Rev. Lett. **4**, 380 (1960); J. Goldstone, Nuovo Cim. **19**, 154 (1961).
[48] X. G. Wen, Phys. Rev. B **44**, 2664 (1991).

[49] J. F. Schonfeld, Nucl. Phys. B **185**, 157 (1981); R. Jackiw and S. Templeton, Phys. Rev. D **23**, 2291 (1981).
[50] R. D. Pisarski, Phys. Rev. D **34**, 3851 (1986); I. Affleck, J. Harvey, L. Palla, and G. Semenoff, Nucl. Phys. B **328**, 575 (1989).
[51] X. G. Wen, F. Wilczek, and A. Zee, Phys. Rev. B **39**, 11413 (1989).
[52] D. Khveshchenko and P. B. Wiegmann, Phys. Lett. B **225**, 279 (1989).
[53] G. E. Volovik and V. M. Yakovenko, J. Phys.: Condens. Matter **1**, 5263 (1989).
[54] M. C. Diamantini, P. Sodano, and C. A. Trugenberger, Nucl. Phys. B **474**, 641 (1996).
[55] T. Dombre and G. Kotliar, Phys. Rev. B **39**, 855 (1989).
[56] S. Liang, B. Doucot, and P. W. Anderson, Phys. Rev. Lett. **61**, 365 (1988).
[57] N. Read and S. Sachdev, Nucl. Phys. B **316**, 609 (1989).
[58] C. Gros, Ann. Phys. **189**, 53 (1989); P. A. Lee, N. Nagaosa, and X. G. Wen, Rev. Mod. Phys. **78**, 17 (2006); M. Ogata and H. Fukuyama, Rep. Prog. Phys. **71**, 036501 (2008).
[59] L. B. Ioffe and A. I. Larkin, Phys. Rev. B **39**, 8988 (1989); N. Nagaosa, Phys. Rev. Lett. **71**, 4210 (1993); I. F. Herbut and B. H. Seradjeh, *ibid.* **91**, 171601 (2003).
[60] Even in the presence of time-reversal symmetry, the mutual (double) Chern-Simon mechanism sometimes protects a given mean-field ansatz from massless gauge fluctuations (Ref. 54). However, we expect that it does not work either here since our BW state belongs to the weak topological insulator—$\mathcal{Z}_2$ even class—(see Sec. IV).
[61] N. Read and S. Sachdev, Phys. Rev. Lett. **62**, 1694 (1989); Phys. Rev. B **42**, 4568 (1990).
[62] Y. Ran, A. Vishwanath, and D.-H. Lee, arXiv:0806.2321 (unpublished).
[63] J. Alicea, Phys. Rev. B **78**, 035126 (2008).
[64] M. U. Ubbens and P. A. Lee, Phys. Rev. B **46**, 8434 (1992).
[65] T. Moriya, *Spin Fluctuations in Itinerant Electron Magnetism* (Springer-Verlag, Berlin, 1985).
[66] R. Shindou and T. Momoi (unpublished).
[67] See, for example, B. Delamotte, D. Mouhanna, and M. Tissier, *Frustrated Spin Systems*, edited by H. T. Diep (World Scientific, Singapore, 2004), Chap. 3; B. Delamotte, D. Mouhanna, and M. Tissier, Phys. Rev. B **69**, 134413 (2004).
[68] M. Sato, Phys. Rev. B **79**, 214526 (2009).